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Internal gravity waves generated by oscillations of a sphere

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We consider the radiation of internal gravity waves from a spherical body oscillating vertically in a stratified incompressible fluid. A near-field solution (under the Boussinesq approximation) is obtained by separation of variables in an elliptic problem, followed by analytic continuation to the frequencies $\omega < N$ of internal wave radiation. Matched expansions are used to relate this solution to a far-field solution in which non-Boussinesq terms are retained. In the outer near field there are parallel conical wavefronts between characteristic cones tangent to the body, but with a wavelength found to be shorter than that for oscillations of a circular cylinder. It is also found that there are caustic pressure singularities above and below the body where the characteristics intersect. Far from the source, non-Boussinesq effects cause a diffraction of energy out of the cones. The far-field wave-fronts are hyperboloidal, with horizontal axes. The case of horizontal oscillations of the sphere is also examined and is shown to give rise to the same basic wave structure.

The related problem of a pulsating sphere is then considered, and it is concluded that certain features of the wave pattern, including the caustic singularities near the source, are common to a more general class of oscillating sources.

1. Introduction

Internal gravity waves are an important feature of oceanic, atmospheric and astrophysical flows. While ray theory provides a good description of the general structure of the wave pattern produced by some disturbance (Lighthill 1978), it is difficult to write down an exact solution giving amplitude information (even for an inviscid case) for the motion resulting from, say, some specified motion of a rigid body.

When the Boussinesq approximation is made, and variation in only two dimensions is assumed, the initial-value problem can be solved using Laplace transforms, which make the governing equation for the pressure elliptic, and complex variable methods can be applied to the spatial problem (Bretherton 1967). Hurley (1972) showed how a steady-state problem could be solved as an elliptic problem for frequency $\omega > N$ and then analytically continued to the hyperbolic case $\omega < N$, where N is the Brunt–Väisälä frequency. The governing equation is also elliptic if rotation is present and the source frequency is sufficiently small. Sarma & Krishna (1972) attempted the general incompressible non-Boussinesq problem for an oscillating spheroid with a solution in terms of spheroidal wave functions, but these are unwieldy and difficult to interpret. Hendershott (1969) considered instead a spherical source beginning to pulsate; the solution was then a function of a radial-type coordinate only. The steady-state, Boussinesq solution was found, but, as we shall show below, singularities were overlooked in the solution when the rotation is small.

While the papers cited above make important contributions to the solution of difficult internal wave boundary-value problems, they leave open questions worthy of further study. For example, although Hurley (1972) gave a general solution in characteristic coordinates that could, in principle, be used to describe the waves generated by some specific motion of a cylinder, he did not in fact obtain the solution for horizontal or vertical oscillations of a rigid cylinder. And with regard to non-Boussinesq effects that might be significant at large distances, we know of no treatment that reveals the essential nature of those effects (although they are formally included in the spheroidal wave function expansion of Sarma & Krishna (1972), but only then in the case of an elliptic problem with rotational effects dominating those of stratification).

To make more specific and detailed predictions about the generation of internal waves by prescribed motions of simple bodies, the authors (Appleby & Crighton 1986) applied a variant of Hurley's (1972) method to the problem of steady-state oscillations of a cylinder, with non-Boussinesq effects included. The exact Boussinesq solution was found as the leading-order term in an inner expansion, and had the familiar 'St. Andrew's Cross' structure (Mowbray & Rarity 1967). This was matched to two far-field expansions, valid in different angular regions, where non-Boussinesq terms were significant. The outer field, insignificant for terrestrial flows, but of more interest in astrophysical and laboratory cases, had a very different structure, with hyperbolic wavefronts, although the energy still remained concentrated near the characteristic surfaces. Non-Boussinesq effects were interpreted as causing the *diffraction* of energy towards the horizontal.

The present paper applies a similar method to the solution of the problem of radiation by a sphere oscillating in an incompressible fluid. A similar complex coordinate transformation is used, and is given only in outline here. The treatment for just the leading-order terms is included; higher-order terms may be derived by the method of matched asymptotic expansions, as in the first paper. We draw attention to certain physical features, which might be investigated experimentally, in which there is a qualitative difference between the fields of a cylinder and a sphere.

Section 2 deals with vertical oscillations of a sphere, §2.1 describing the coordinate transformation and the Boussinesq, or leading-order inner, solution, and §2.2 the outer solutions valid at large distances from the source. For a vertically oscillating source, there is no azimuthal variation and the solution is described in terms of Legendre functions matched to spherical Bessel functions. In §2.3 these results are interpreted for the radiating case $\omega < N$. An indication of the general structure of the solution for the case of horizontal oscillations is given in §3. Hendershott's pulsating spherical source is considered in §4. In the discussion of §5 these three model solutions are compared and contrasted and some general conclusions are drawn concerning the wave structure and the range of validity of these results.

2. Vertical oscillations of a sphere

2.1. *The problem for $\omega > N$*

An exponential density profile $\tilde{\rho}_0 = \tilde{\rho}_{00}\exp(-\beta z)$ is assumed, giving a constant Brunt–Väisälä frequency $N = (g\beta)^{\frac{1}{2}}$. Then the pressure p is written as $\exp(-\frac{1}{2}\beta z)\,q$, and assumed to be a function only of the radial and vertical cylindrical coordinates r and z, and of time through the factor $\exp(-\mathrm{i}\omega t)$ which is suppressed in the following. The coordinates are made dimensionless with the radius of the sphere.

Then the equations of axisymmetric (non-swirling) inviscid incompressible flow under gravity g can, in the linear approximation, be combined to give

$$\frac{\partial^2 q}{\partial z^2}-\left(\frac{N^2-\omega^2}{\omega^2}\right)\left\{\frac{\partial^2 q}{\partial r^2}+\frac{1}{r}\frac{\partial q}{\partial r}\right\}-\tfrac{1}{4}\beta^2 q = 0; \tag{2.1}$$

see Sarma & Krishna (1972). If the vertical velocity amplitude of the sphere is taken as unity, the inviscid boundary condition is

$$ur+wz = z \quad \text{on} \quad r^2+z^2 = 1, \tag{2.2}$$

where (u, w) are the velocity components corresponding to (r, z).

For $\omega > N$, (2.1) is elliptic, and a stretched oblate spheroidal transformation may be applied:

$$r = \alpha c\,(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}, \quad z = c\xi\eta, \tag{2.3}$$

where

$$\alpha = \left[\frac{\omega^2-N^2}{\omega^2}\right]^{\frac{1}{2}}, \quad c = \left[\frac{N^2}{\omega^2-N^2}\right]^{\frac{1}{2}}.$$

Then (2.1) and (2.2) become

$$(1+\xi^2)\frac{\partial^2 Q}{\partial \xi^2}+2\xi\frac{\partial Q}{\partial \xi}+(1-\eta^2)\frac{\partial^2 Q}{\partial \eta^2}-2\eta\frac{\partial Q}{\partial \eta} = \tfrac{1}{4}\beta^2 c^2(\xi^2+\eta^2)\,Q, \tag{2.4}$$

with

$$\frac{\partial Q}{\partial \xi}-\beta\frac{(1-\alpha^2)^{\frac{1}{2}}}{2\alpha}\eta Q = \eta\exp(1-\tfrac{1}{2}\beta\eta) \quad \text{on} \quad \xi = \xi_0 = \frac{1}{c}, \tag{2.5}$$

and where

$$q = \mathrm{i}\tilde{\rho}_{00}N\alpha Q$$

defines Q.

Full solutions to (2.4) can be described in terms of spheroidal wave functions, but if β (which is the ratio of the sphere radius to the atmospheric density scale height) is small, the leading-order solution (in an expansion in powers of β) satisfies Legendre's equation in each coordinate. Near to the source, i.e. for $(\xi, \eta) = O(1)$ as $\beta \to 0$, we write $\tilde{Q}(\xi, \eta)$ for the inner solution, which must be of the form

$$\tilde{Q} \sim P_n(\mathrm{i}\xi)\,P_n(\eta) \quad \text{or} \quad Q_n(\mathrm{i}\xi)\,P_n(\eta),$$

since solutions $Q_n(\eta)$ would imply logarithmic variation in the 'angular' coordinate and would not be periodic. At large distances R from the source, $|\mathrm{i}\xi|$ is large, and although the pressure perturbation need not be small, the velocity perturbation should certainly decay. Hence the $P_n(\mathrm{i}\xi)$ are also unacceptable.

For the leading term,

$$\frac{\partial \tilde{Q}}{\partial \xi} = \eta \quad \text{on } \xi = \xi_0, \tag{2.6}$$

and hence

$$\tilde{Q} = A_0 Q_1(\mathrm{i}\xi) P_1(\eta) + O(\beta), \tag{2.7}$$

with

$$A_0 \frac{\partial}{\partial \xi} Q_1(\mathrm{i}\xi) = 1 \quad \text{on } \xi = \xi_0. \tag{2.8}$$

Now

$$Q_1(\mathrm{i}\xi) = \tfrac{1}{2}\mathrm{i}\xi \ln\left(\frac{\mathrm{i}\xi+1}{\mathrm{i}\xi-1}\right) - 1 = \xi \tan^{-1}\left(\frac{1}{\xi}\right) - 1,$$

and therefore, for large values of ξ, the behaviour of the leading-order pressure field is given by

$$\tilde{Q} \sim A_0\left(-\frac{1}{3\xi^2}\right)\eta = O(R^{-2}), \tag{2.9}$$

with algebraic decay. Since $(\tilde{Q}/A_0)$ is purely real it follows that p and u are $\frac{1}{2}\pi$ out of phase, implying no energy radiation. At this level of approximation, the inner solution satisfies equations in which the Boussinesq approximation has been made.

The inner solution continues (as in Appleby & Crighton 1986) with a term of order β, forced by the boundary condition (2.5). Conventional treatments considering only the *propagation* of internal waves claim that all first-order effects in β are accounted for by the factor $\exp(-\frac{1}{2}\beta z)$ extracted from p and by the factor $\exp(+\frac{1}{2}\beta z)$ extracted similarly from all fluctuating velocities, and that when these factors have been extracted the relative error in the Boussinesq approximation is $O(\beta^2)$. This is evidently inadequate to take care of all $O(\beta)$ effects if any boundary condition on a finite body is to be imposed. Similar considerations apply in other wave generation problems – for example, acoustic radiation in the nearly incompressible limit.

We should note here that the inner solutions do not depend on an exponential density distribution. As long as the density profile has a scale height H large compared with the body scale a, the same inner solutions will apply throughout $|\boldsymbol{x}| \ll H$, and these solutions are therefore particularly important. On scales $|\boldsymbol{x}| = O(H)$, however, the detailed nature of the profile becomes important, and the specific results for the outer solutions therefore apply only to the exponential profile – though one may reasonably anticipate that the qualitative effects seen at large distances for this profile will also be of general relevance.

2.2. *Outer solutions*

Unless the Boussinesq approximation ($\beta = 0$) has been made, the term $\frac{1}{4}\beta^2\xi^2$ in (2.4) necessarily becomes significant at large distances from the source. To take account of this, let $\mu = \frac{1}{2}c\beta\xi$, and write $\hat{Q}$ for the outer solution; then (2.4) becomes

$$\mu^2 \frac{\partial^2 \hat{Q}}{\partial \mu^2} + 2\mu \frac{\partial \hat{Q}}{\partial \mu} - \mu^2 \hat{Q} + (1-\eta^2)\frac{\partial^2 \hat{Q}}{\partial \eta^2} - 2\eta \frac{\partial \hat{Q}}{\partial \eta} = \tfrac{1}{4}\beta^2 c^2 \left\{\eta^2 \hat{Q} - \frac{\partial^2 \hat{Q}}{\partial \mu^2}\right\}. \tag{2.10}$$

The leading-order solution will be a product of a modified spherical Bessel function with a Legendre function, and choosing these to give decay with distance, we have

$$\hat{Q} \sim \beta^m B_0 \left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(\mu) P_n(\eta). \tag{2.11}$$

This must be matched to $\tilde{Q}$ to determine m, n, B_0 in a region where ξ is large but μ small. The η-dependence of $\tilde{Q}$ implies that $n = 1$; then for small μ,

$$\hat{Q} \sim \beta^m B_0 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Gamma(\tfrac{3}{2})\, 2^{\frac{1}{2}} \frac{1}{\mu^2} \eta = \beta^{m-2} B_0 \frac{2\pi}{c^2} \frac{1}{\xi^2} \eta,$$

while for large ξ, (2.9) gives

$$\tilde{Q} \sim -\frac{A_0}{3} \frac{1}{\xi^2} \eta.$$

Therefore $m = 2$ and

$$B_0 = -\frac{1}{6\pi} A_0 c^2,$$

so that

$$\hat{Q} \sim -\beta^2 \frac{A_0 c^2}{6\pi} \left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}} K_{\frac{3}{2}}(\mu) P_1(\eta). \tag{2.12}$$

At large distances from the source, this $\hat{Q}$ is of the form

$$R^{-1} \exp(-kR),$$

with $k \equiv k(\eta)$ constant along any given cone with vertical axis. The pressure and (spherical) radial velocity fluctuations are again out of phase and, as expected since we still have $\omega > N$, no energy is radiated to infinity.

From Appleby & Crighton (1986) we anticipate the need for an additional representation $\bar{Q}$ when η is large, although this is redundant in the case $\omega > N$. This representation, and the corresponding asymptotic solution, refers to imaginary angles when $\omega > N$ (when it is redundant), but to real angles when analytic continuation is made to frequencies $\omega < N$.

Let $\lambda = \frac{1}{2}c\beta\eta$; then (2.10) becomes

$$\left[\mu^2 \frac{\partial^2 \bar{Q}}{\partial \mu^2} + 2\mu \frac{\partial \bar{Q}}{\partial \mu} - \mu^2 \bar{Q}\right] - \left[\lambda^2 \frac{\partial^2 \bar{Q}}{\partial \lambda^2} + 2\lambda \frac{\partial \bar{Q}}{\partial \lambda} + \lambda^2 \bar{Q}\right] = -\tfrac{1}{4}\beta^2 c^2 \left[\frac{\partial^2 \bar{Q}}{\partial \mu^2} + \frac{\partial^2 \bar{Q}}{\partial \lambda^2}\right]. \tag{2.13}$$

The leading-order solution as $\beta \to 0$ for $(\lambda, \mu) = O(1)$ will be a product of spherical Bessel functions. From the form of $\hat{Q}$ and the need to avoid non-periodic angular dependence, we choose

$$\bar{Q} = \beta^s C_0 \left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(\mu) \left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda) + O(\beta^{s+1}). \tag{2.14}$$

For λ small, η large, this should match (2.12), so $n = 1$ is taken. For λ small,

$$\left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(\lambda) \approx \tfrac{1}{4}\pi^{\frac{1}{2}} \frac{1}{\Gamma(\frac{5}{2})} \lambda$$

and hence $s = 1$ and $C_0 = -A_0 c/\pi$. Therefore we have

$$\bar{Q} \sim -\beta \left(\frac{A_0 c}{\pi}\right) \left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}} K_{\frac{3}{2}}(\mu) \left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(\lambda). \tag{2.15}$$

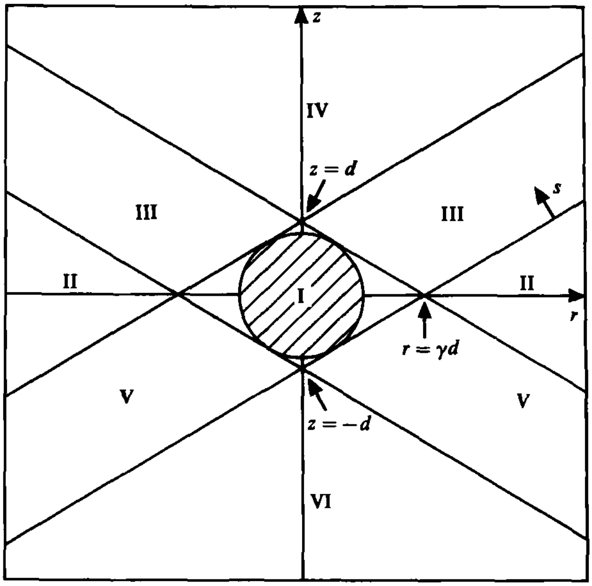

FIGURE 1. Diametral section through the oscillating sphere. Region III is a section through the upper characteristic cone, region V a section through the lower characteristic cone. In the Boussinesq approximation, wave activity is confined to III and V. Diffraction into region II causes concentrations of wave energy near the lower boundary of III and near the upper boundary of V. Focusing singularities appear at $z = \pm d$, $r = 0$.

When analytic continuation is made to the case $\omega < N$, the expressions (2.12) and (2.15) are both needed to describe the (non-Boussinesq) far field, but in different regions, as described in §2.3.

2.3. *Analytic continuation to $\omega < N$*

For $\omega < N$, α becomes imaginary, and we write $\alpha = \mathrm{i}\gamma$, $c = -\mathrm{i}d$, giving

$$r = \gamma d(1+\xi^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}, \quad z = -\mathrm{i}d\xi\eta, \tag{2.16}$$

so that ξ, η are now *complex* coordinates. Real (r, z)-space can be divided into six regions separated by conical characteristic surfaces, in which ξ, η have different complex components (figure 1). Writing

$$\xi = \sinh\rho, \quad \eta = \sin\theta; \qquad \rho = \sigma + \mathrm{i}\tau, \quad \theta = \kappa + \mathrm{i}\nu, \tag{2.17}$$

then in region II, for example, ρ is real and θ is imaginary. In region I, θ is real and ρ is imaginary. Also $\mathrm{Re}\,(\xi)$, $\mathrm{Im}\,(\xi) \geqslant 0$ always. A fuller description of the use of these complex coordinates can be found in Appleby & Crighton (1986).

Consider first the inner, or Boussinesq, solution $\tilde{Q}$, given by (2.7) and (2.8). The points $r = 0$, $z = \pm d$ are on the boundaries between regions I, III, IV and I, V, VI respectively. Here $\tau = \frac{1}{2}\pi$, $\sigma = \gamma = 0$, $\kappa = \pm\frac{1}{2}\pi$ and $\xi = \mathrm{i}$, so that there is a logarithmic singularity in the pressure at these points. Since $\xi = \mathrm{i}$ at *both* points, there is no branch cut in real space, and the logarithmic term may be defined uniquely by taking a real value on the z-axis at large distances from the source (A_0 must be expressed as a logarithm rather than as an inverse tangent for $\omega < N$). For $\omega > N$ these singularities do not arise. Such singularities have not previously been encountered in internal wave problems.

Further from the source, the behaviour is given by (2.9). In region III, $\sigma = \nu$ and both are large together, so

$$\begin{aligned}\tilde{Q} &\sim -\tfrac{1}{3}A_0\, 2\mathrm{i}\mathrm{e}^{\nu-2\sigma}\,\mathrm{e}^{-\mathrm{i}(\kappa+2\tau)}\\ &= -\tfrac{2}{3}\mathrm{i}A_0\,\mathrm{e}^{-\sigma}\,\mathrm{e}^{-3\mathrm{i}\tau}\\ &\sim R^{-\frac{1}{2}}\mathrm{e}^{-3\mathrm{i}\tau},\end{aligned} \tag{2.18}$$

which shows that there is phase variation only across the cone III. Since $\cos^2\tau$ varies linearly from 1 to 0 across this conical region, this implies a shorter wavelength than that for the oscillating cylinder, where $\tilde{Q} \sim \exp(-2\mathrm{i}\tau)$ (Appleby & Crighton 1986). Region V has similar behaviour.

In regions II, IV and VI, κ, τ are constant, and ν is asymptotically constant on any cone through the origin, so that

$$\tilde{Q} \sim R^{-2}$$

radially, and is of constant phase implying, as anticipated, that no energy is radiated in these regions.

The pattern overall is therefore similar to that for the well-known two-dimensional case, although with waves of shorter wavelength. Internal waves with wavefronts parallel to a characteristic and with a two-dimensional spreading factor propagate within two conical regions, and there are non-radiating disturbances elsewhere.

For the outer solution

$$\begin{aligned}\hat{Q} &\sim \frac{1}{\mu^{\frac{1}{3}}}K_{\frac{2}{3}}(\mu)\sin\theta,\\ &\sim \frac{1}{\mathrm{e}^{\sigma+\mathrm{i}\tau}}\exp\{\tfrac{1}{4}d\beta\mathrm{e}^{\sigma}(-\sin\tau+\mathrm{i}\cos\tau)\}\sin\theta, \quad \text{(at large distances)}\\ &\sim \frac{1}{\mathrm{e}^{\sigma+\mathrm{i}\tau}}\mathrm{e}^{\nu-\mathrm{i}\kappa}\exp\{\tfrac{1}{4}d\beta\mathrm{e}^{\sigma}(-\sin\tau+\mathrm{i}\cos\tau)\},\end{aligned} \tag{2.19}$$

if $\sin\theta$ is large also, but such that $\beta\sin\theta$ is still small (otherwise $\bar{Q}$ must be taken). Then in region II, $\kappa = \tau = 0$, and on any radial cone,

$$\hat{Q} \sim \frac{1}{R}\exp\{\mathrm{i}kR\}, \tag{2.20}$$

with $k \equiv k(\nu)$. This describes *spherically* diverging waves propagating at angles to the vertical greater than the characteristic angle, and with hyperboloidal wavefronts (σ = constant).

When μ, η are both large, $\bar{Q}$ must be taken, and

$$\begin{aligned}\bar{Q} &\sim \frac{1}{\mu^{\frac{1}{3}}}\frac{1}{\lambda^{\frac{1}{3}}}K_{\frac{2}{3}}(\mu)J_{\frac{2}{3}}(\lambda),\\ &\sim \frac{1}{\mathrm{e}^{\sigma+\mathrm{i}\tau}}\frac{1}{\mathrm{e}^{\nu-\mathrm{i}\kappa}}\exp\{\tfrac{1}{4}d\beta\mathrm{e}^{\sigma}(\mathrm{i}\cos\tau-\sin\tau)\}\exp\{\tfrac{1}{4}d\beta\mathrm{e}^{\nu}(\mathrm{i}\cos\kappa+\sin\kappa)\},\end{aligned} \tag{2.21}$$

with behaviour like that of $\hat{Q}$ in region II.

In region III (and similarly in region V) parallel to the characteristics,

$$\bar{Q} \sim \frac{1}{R}\exp(\mathrm{i}kR^{\frac{1}{2}}), \tag{2.22}$$

with $k \equiv k(\tau)$. Here the wavefronts follow no familiar pattern, but (2.22) describes the diffraction process as conical wavefronts give way to hyperboloidal wavefronts. As distance from the source increases, there is progressively less energy propagated within regions III and V, as diffraction concentrates the wave energy close to the characteristic cone nearer the horizontal and causes wave energy also to be propagated – in spherical wave form – in region II. As the other cones are approached (i.e. those nearer the vertical), the radial wavelength becomes infinite.

In regions IV and VI, $\tau = \frac{1}{2}\pi$, and both $\hat{Q}$ and $\bar{Q}$ are of the form

$$\frac{1}{R}\exp(-k(\nu)R),$$

with exponential decay and no energy propagation.

3. Horizontal oscillations of a sphere

If the sphere oscillates horizontally, the solution will vary azimuthally. There will be also an induced mean flow, although this will not be described by the linearized equations used here. The scaled pressure, as defined in §2, now obeys

$$\frac{\partial^2 Q}{\partial z^2}+\alpha^2\left\{\frac{\partial^2 Q}{\partial r^2}+\frac{1}{r}\frac{\partial Q}{\partial r}\right\}+\frac{\alpha^2}{r^2}\frac{\partial^2 Q}{\partial \varphi^2}-\tfrac{1}{4}\beta^2 Q = 0, \tag{3.1}$$

where φ is the azimuthal angle, and the boundary condition becomes

$$ur+wz = r\cos\varphi \quad \text{on } r^2+z^2 = 1. \tag{3.2}$$

Transformed as before, these become

$$(1+\xi^2)\frac{\partial^2 Q}{\partial \xi^2}+2\xi\frac{\partial Q}{\partial \xi}+(1+\eta^2)\frac{\partial^2 Q}{\partial \eta^2}-2\eta\frac{\partial Q}{\partial \eta}+\left[\frac{1}{(1-\eta^2)}-\frac{1}{(1+\xi^2)}\right]\frac{\partial^2 Q}{\partial \varphi^2} = \tfrac{1}{4}\beta^2 c^2(\eta^2+\xi^2)Q, \tag{3.3}$$

and

$$\frac{\partial Q}{\partial \xi}-\frac{\beta(1-\alpha^2)^{\frac{1}{2}}}{2\alpha}\eta Q = (1-\eta^2)^{\frac{1}{2}}\cos\varphi\exp(-\tfrac{1}{2}\beta\eta) \quad \text{on } \xi = \xi_0. \tag{3.4}$$

The complete solution can be described by oblate spheroidal wave functions, but we prefer to proceed as before using asymptotics as $\beta \to 0$. The leading-order inner solution satisfies the double associated Legendre equation given by the left-hand side of (5.3), with the boundary condition

$$\frac{\partial \tilde{Q}}{\partial \xi} = (1-\eta^2)^{\frac{1}{2}}\cos\varphi \quad \text{on } \xi = \xi_0. \tag{3.5}$$

This gives the solution $\tilde{Q} = \tilde{Q}_0+O(\beta)$, with

$$\begin{aligned}\tilde{Q}_0 &= AQ_1^1(i\xi)P_1^1(\eta)\cos\varphi \\ &= -A(1+\xi^2)^{\frac{1}{2}}\left[\tfrac{1}{2}\ln\left(\frac{1+i\xi}{1-i\xi}\right)+\left(\frac{i\xi}{1+\xi^2}\right)\right](1-\eta^2)^{\frac{1}{2}}\cos\varphi,\end{aligned} \tag{3.6}$$

where A is given by

$$A\frac{\mathrm{d}}{\mathrm{d}\xi}Q_1^1(\mathrm{i}\xi) = 1 \quad \text{on} \quad \xi = \xi_0. \tag{3.7}$$

The structure of this solution is very similar to that for vertical oscillations away from the source, but for the variation with azimuthal angle φ. However, the logarithmic singularities at $z = \pm d$ $(\xi = \mathrm{i})$ are now dominated by *algebraic singularities*. The pressure varies as $(\xi-\mathrm{i})^{-\frac{1}{2}}$, the velocity as $(\xi-\mathrm{i})^{-\frac{3}{2}}$, so that the energy flux into a vanishingly small volume around the points $z = \pm d$ is finite, and the energy in such a small volume is infinite, like $\ln\delta$ where δ is the radius of the volume.

For $\beta\xi$ large, let $\mu = \frac{1}{2}\beta c\xi$; then (3.3) becomes

$$\mu^2\frac{\partial^2\hat{Q}}{\partial\mu^2} + 2\mu\frac{\partial\hat{Q}}{\partial\mu} - \mu^2\hat{Q} + (1-\eta^2)\frac{\partial^2\hat{Q}}{\partial\eta^2} - 2\eta\frac{\partial\hat{Q}}{\partial\eta} + \frac{1}{1-\eta^2}\frac{\partial^2\hat{Q}}{\partial\varphi^2} = O(\beta^2). \tag{3.8}$$

The leading-order solution for $\hat{Q}$, describing the outer wave field, will be a product of a modified spherical Bessel function and an associated Legendre function, and the analysis proceeds much as before.

4. The pulsating spherical source

Hendershott (1969) used a simpler boundary condition on the sphere, namely that of fixed (oscillating) normal velocity. In our coordinates this is

$$\frac{\partial Q}{\partial\xi} - \frac{\beta(1-\alpha^2)^{\frac{1}{2}}}{2\alpha}\eta Q = \exp\left(-\tfrac{1}{2}\beta\eta\right) \quad \text{on} \quad \xi = \xi_0. \tag{4.1}$$

The Boussinesq solution satisfies

$$\frac{\partial\tilde{Q}}{\partial\xi} = 1 \quad \text{on} \quad \xi = \xi_0, \tag{4.2}$$

and is

$$\tilde{Q} = A_0 Q_0(i\xi) = A_0\tfrac{1}{2}\ln\left(\frac{\mathrm{i}\xi+1}{\mathrm{i}\xi-1}\right), \tag{4.3}$$

with logarithmic singularities in the pressure. Away from the source, in the wave regions III and V,

$$\tilde{Q} \sim \frac{1}{R^{\frac{1}{2}}}\mathrm{e}^{-\mathrm{i}\tau}, \tag{4.4}$$

so that the phase variation across regions III and V is less rapid than before; this agrees with Hendershott's solution.

For $\beta\xi$ large we readily find the non-Boussinesq outer solution

$$\hat{Q} \sim -\beta\left(\frac{2\mathrm{i}cA_0}{\pi}\right)\left(\frac{\pi}{2\mu}\right)^{\frac{1}{2}}K_{\frac{1}{2}}(\mu), \tag{4.5}$$

with a structure asymptotically the same as that for the oscillating sphere. $\bar{Q}$ may also be derived for this case.

In Hendershott's analysis, where solid-body rotation with angular velocity Ω was included, the logarithmic singularities for the Boussinesq solution do not appear for $\Omega^2 > N^2$, but *are* present (and are overlooked) in the case $N^2 > \Omega^2$, at the same points, above and below the sphere, at the intersection of the characteristic cones. To see this, consider Hendershott's equation (22) for the vertical velocity,

$$w(r,z,t) = \frac{z}{2\pi \mathrm{i}(r^2+z^2)} \int_{Br} \mathrm{e}^{st} \frac{s\bar{U}(s)\,\mathrm{d}s}{(s^2+\Sigma_+^2)^{\frac{1}{2}}(s^2+\Sigma_-^2)^{\frac{1}{2}}}, \tag{4.6}$$

where Br is the usual Bromwich inversion path for Laplace transforms, $\bar{U}(s)$ is the Laplace transform of the radial velocity of the sphere, and in Hendershott's (22) we have put the sphere radius $a = 1$ and the rotation parameter $f = 0$. With the same replacements, Hendershott's (21) defines the $\Sigma_\pm$ as

$$\Sigma_\pm = N\left(\frac{z(r^2+z^2-1)^{\frac{1}{2}} \mp r}{r^2+z^2}\right), \tag{4.7}$$

so that on the axis $r = 0$, $\Sigma_+ = \Sigma_-$, and the branch points of the integrand coalesce to form simple poles at $s = \pm \mathrm{i}\Sigma$. Now if the source velocity is $\exp(-\mathrm{i}\omega t)$ for $t > 0$, then $\bar{U}(s) = 1/(s+\mathrm{i}\omega)$ and the integrand of (4.6) has simple poles at $\pm \mathrm{i}\Sigma$ and at $s = -\mathrm{i}\omega$. However, if we choose the point on $r = 0$ to coincide with the upper or lower apex of the characteristic cone for frequency ω ($z = \pm d$ in figure 1), then geometry shows that $\Sigma = -\omega$, and $s = -\mathrm{i}\omega$ is a double pole. At these points, therefore, w grows with time like $t \exp(-\mathrm{i}\omega t)$ and there is no steady state with finite amplitude at these points ($r = 0$, $z = \pm d$). At points ($r = 0$, $z = \pm d + \epsilon$) there is finite steady-state response at the forcing frequency, and the amplitude (of w) is $O((\Sigma+\omega)^{-1}) = O(\epsilon^{-1})$. The pressure singularity is therefore $O(\ln \epsilon)$, as predicted by (4.3), and it can be verified that the coefficients of the singular terms predicted by (4.3) and by (4.6) agree. The fact that these singularities were overlooked by Hendershott in no way invalidates the analysis of his paper which, in the main, is concerned with transient and steady-state response at distances large compared with the sphere radius but (as Hendershott makes clear) small compared with the scale height, $1 \ll (r^2+z^2)^{\frac{1}{2}} \ll \beta^{-1}$. A local analysis of the singular regions (to prove that the singularities do indeed have only local significance) would be interesting, but is not attempted here.

5. Discussion and conclusions

Three related problems concerning the generation of internal gravity waves by oscillating bodies have been considered in this and a preceding paper (Appleby & Crighton 1986). In each case a solution (the *inner* solution) was found under the Boussinesq approximation, valid out to large distances except in some astrophysical or laboratory situations. The wave structure was found to be similar in each case, having parallel, unchanging wavefronts between characteristic surfaces tangent to the body, although with different wavelengths. The form of the three Boussinesq solutions and the corresponding outward (i.e. parallel to the characteristics) velocity components in region III are summarized in table 1. If we let $\cos 2\tau = s$, then s varies linearly from 1 to 0 going from region II to region IV, and the phase structure can be written:

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}\tau} = [(1+s)^{\frac{1}{2}} - \mathrm{i}(1-s)^{\frac{1}{2}}]/\sqrt{2} \quad \text{(pulsating sphere)},\\
&\mathrm{e}^{-2\mathrm{i}\tau} = s - \mathrm{i}(1-s^2)^{\frac{1}{2}} \quad \text{(oscillating cylinder)},\\
&\mathrm{e}^{-3\mathrm{i}\tau} = [(1+s)^{\frac{1}{2}}(2s-1) - \mathrm{i}(1-s)^{\frac{1}{2}}(2s+1)]\sqrt{2} \quad \text{(oscillating sphere)}.
\end{aligned}$$

Source	$q(\rho, \theta)$ pressure	Wave-field structure	Outward velocity	Wavelengths across region
Pulsating sphere	$e^{-\rho}$	$\dfrac{1}{R^{\frac{1}{2}}}e^{-i\tau}$	$\dfrac{e^{-i\tau}}{R^{\frac{1}{2}}\sin 2\tau}$	$\frac{1}{4}$
Oscillating cylinder	$e^{-\rho}\sin\theta$	$e^{-2i\tau}$	$\dfrac{e^{-2i\tau}}{\sin 2\tau}$	$\frac{1}{2}$
Oscillating sphere	$e^{-2\rho}\sin\theta$	$\dfrac{1}{R^{\frac{1}{2}}}e^{-3i\tau}$	$\dfrac{e^{-3i\tau}}{R^{\frac{1}{2}}\sin 2\tau}$	$\frac{3}{4}$

TABLE 1. Principal features of Boussinesq (inner) solutions in region III of figure 1

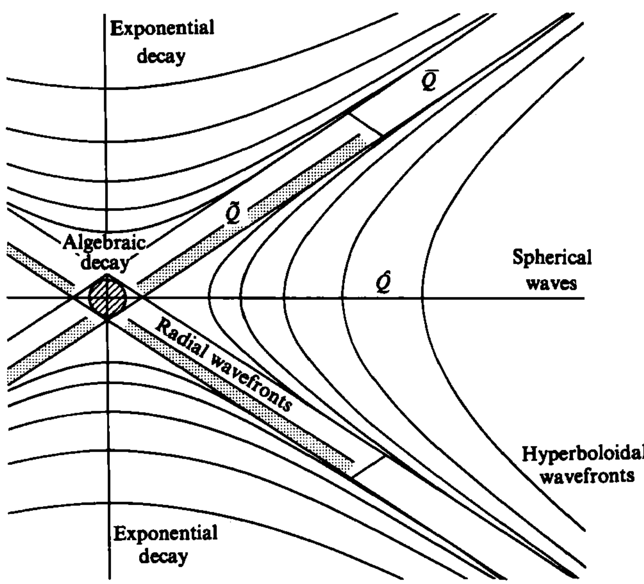

FIGURE 2. Regions of different wave structure and asymptotic expansions. The shading is intended to suggest the radial wavefronts in the case of vertical oscillations.

This illustrates the difficulty of working in more comprehensible coordinates. Note also that

$$\int_0^1 \frac{1}{\sin 2\tau}\, d(\cos 2\tau) = -2$$

and hence the velocity singularities at the edges of the characteristic cones are *integrable* and are local features of the inviscid formulation. The integrated energy flux is finite in each case.

In all cases, non-Boussinesq effects in the far field lead to a diffraction of energy from the characteristic regions to give hyperboloidal (or hyperbolic) wavefronts between these regions and the horizontal, although most of the energy is concentrated near the characteristics (see figure 2).

For the two cases with a spherical source, singularities appear in the pressure (and hence in the velocity) at points above and below the source, where the characteristic cones intersect. These may be termed *caustic* or *focusing singularities*. Although in a real fluid viscosity would make the perturbation finite here, small regions of intense activity should be observed in an experiment. More intense fluctuations should arise for horizontal oscillations, where the singularities are algebraic, than with vertical, where they are logarithmic. Experiments with spheres, rather than the more commonly used circular cylinders, are needed to see if this qualitative prediction is borne out.

These singularities correspond to a finite level of activity in the wave field effectively generated by the perturbation at a point. In the two-dimensional case the characteristics intersect on an infinite line and these singularities do not occur (although the velocity becomes infinite at the edges of the wave regions). For any three-dimensional body such that part of the tangential characteristic surfaces is in the form of a circular cone there will, on the other hand, be such a concentration of activity. In general, it is to be expected that small regions of intense activity will be observed in the neighbourhood of any oscillating three-dimensional body, although oscillations of large amplitude would spread this area more widely. In particular, ellipsoidal bodies should exhibit essentially the same singularities, as well as similar wave structure. These could be treated by the same analytical method.

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